

Proposition 0.1. *The set $\Phi := \Sigma_1 \cup \{xyt_1xt_2y \approx yxt_1xt_2y, xytxy \approx xytyx\}$ is a monoid basis for the identities of $S(\{asbtab\})$.*

Proof. Let w and w' be words for which $S(\{asbtab\}) \models w \approx w'$. We show that $\Phi \vdash w \approx w'$. The identity $xt_1xt_2x \approx xxt_1t_2 \in \Phi$ can be used to reduce every word to one that is 2-limited. Thus it suffices to consider the case where both w and w' are 2-limited.

We now make an observation that is used throughout the proof.

Observation. In the presence of Φ , the first occurrence of a 2-occurring letter in a word commutes with any occurrence of any neighbouring 2-occurring letter. That is:

$$\Phi \vdash ux_1yv \approx u_1yxv \quad (\text{where } x \text{ is 2-occurring}).$$

This follows easily from the identities $xysxty \approx yxsxty$ and $xsxyty \approx xsyxyt$ and their deletions.

Let x be a 2-occurring letter in a word w for which there is no linear letter t such that $w(x, t) \equiv xtx$ and let w_x be the word obtained from w by deleting x . We now show that $\Phi \vdash w \approx xxw_x$. If xx is a subword of w , then this claim follows immediately from $xxt_1 \approx t_1xx \in \Phi$. If xx is not a subword then we may write w in the form $w_1xyw_2xw_3$ where y is 2-occurring in w and w_2 is a possibly empty subword containing no letters that are linear in w . Using the observation we may derive $w \approx w_1xw_2xw_3$. Repeating this eventually achieves $\Phi \vdash w \approx w_1w_2xxw_3$ and then apply $xxt_1 \approx t_1xx \in \Phi$ as before. Thus $\Phi \vdash w \approx xxw_x$. Now x is also 2-occurring in w' , and as xtx is an isoterm, the same argument on w' achieves $\Phi \vdash w' \approx xxw'_x$. This shows that it suffices to prove $\Phi \vdash w \approx w'$ under the assumption that every 2-occurring letter in w (whence w') has an occurrence of a linear letter between its first and second occurrence.

By repeating the above paragraph for each 2-occurring letter that does not have occurrences either side of a linear letter, we eventually arrive at a word of the form uv , where u is a product of squares of letters (that can be arbitrarily commuted using $xxt_1 \approx t_1xx \in \Phi$) and every 2-occurring letter in v occurs either side of a linear letter. Now say that $w \approx w'$ is a 2-limited identity satisfied by $S(\{asbtab\})$, where both w and w' have been rearranged in this way; that is $w \equiv uv$ and $w' \equiv u'v'$, where u and u' are products of squares of letters and every two occurring letter in v and v' occurs either side of a linear letter. By deleting letters it is clear that $S(\{asbtab\})$ also satisfies $u \approx u'$ and $v \approx v'$. Because we can commute squares using Φ , we find that $\Phi \vdash u \approx u'$. Therefore it will now suffice to show that $\Phi \vdash v \approx v'$.

Because xyx is an isoterm for $S(\{asbtab\})$, any unstable pair in $v \approx v'$ must involve only 2-occurring letters. Let (i, j) be a critical pair. We show how to remove (i, j) from the set of unstable pairs in $v \approx v'$, without adding any new unstable pairs. While the set of critical pairs will change after an application of this move, the set of unstable pairs is finite, and so repeated applications of the argument eventually eliminate all unstable pairs, thus yielding a proof that $\Phi \vdash v \approx v'$.

There are several possibilities for the pattern of occurrences of x and y in v . However the Observation shows that the critical pair may be removed, except possibly in the case where both i and j are 2. Thus we now consider the case where v is of the form $v_1xv_2yv_3xyv_4$ or $v_1yv_2xv_3xyv_4$. As we are going to show that $\Phi \vdash v_1xv_2yv_3xyv_4 \approx v_1xv_2yv_3yxyv_4$ (with the right hand side of the same form as the $v_1yv_2xv_3xyv_4$ up to a change of letter names), it suffices to consider the case $v_1xv_2yv_3xyv_4$ only.

The assumption that a linear letter occurs between the two occurrences of every 2-occurring letter forces v_3 to contain a linear letter. Then it also follows that v_2 contains only letters that are 2-occurring in v (as $xsytxy$ is an isoterm). Now we may use the Observation to deduce

$$\Phi \vdash v_1xv_2yv_3xyv_4 \approx v_1xyv_2v_3xyv_4 \approx v_1xyv_2v_3yxv_4 \approx v_1xv_2yv_3yxv_4$$

as required: the pair $({}_2x, {}_2y)$ is no longer unstable, and no new unstable pairs are created. Thus we can achieve the deduction of $v \approx v'$ as described. ■

Example 0.2. *The monoid variety defined by $\Sigma_1 \cup \{xyt_1xt_2y \approx yxt_1xt_2y\}$ is generated by $\{S(\{asbt\}), \text{Syn}_M(\{absab, basab\})\}$.*

Proof. Note that $\text{Syn}_M(\{absab, basab\})$ is the quotient of $S(\{absab, basab\})$ by identifying the two elements $absab$ and $basab$, which should coincide in the presence of the identity $xytxy \approx yxtxy$. It is routine from here to show that $\text{Syn}_M(\{absab, basab\})$ satisfies the basis, so that the two semigroups together generate at least a subvariety of that defined by the given identities. We now show that the identities are a basis. The argument is essentially that of the previous proof. The arguments are identical until we arrive at the restriction to the case where x, y are 2-occurring letters for which $({}_2x, {}_2y)$ is unstable. However now this case simply cannot occur: recall that v_3 contained a letter that is linear in v , so that v deletes to $xytxy$. While this is not an isoterm for the variety (as $xytxy \approx yxtxy$ holds), it is easy to verify that $({}_2x, {}_2y)$ is stable in all satisfied identities $xytxy \approx u$. ■